JOURNAL OF FUNCTIONAL ANALYSIS **143**, 33–41 (1997)
ARTICLE NO. FU962963

Restrictions of Unitary Representations to Lattices and Associated C^*-Algebras

M. B. Bekka

Département de Mathématiques, Université de Metz, Ile du Saulcy, F-57045 Metz, France

Received May 4, 1995; accepted November 28, 1995

Let G be a connected simple Lie group with finite centre, and let Γ be a lattice in G. Let π be an irreducible unitary representation of G, and let $\pi \mid \Gamma$ be the restriction of π to Γ. A result by Cowling and Steger states that $\pi \mid \Gamma$ remains irreducible if π is not a discrete series representation and that $\pi \mid \Gamma$ is determined by π. Our first result shows that even the weak equivalence class of $\pi \mid \Gamma$ is determined by π when π is a complementary series representation. Let $C^*_\pi(\Gamma)$ denote the C^*-algebra generated by all $\pi(\gamma)$ for γ in Γ. We show that $C^*_\pi(\Gamma)$ has a unique maximal two-sided ideal and a unique normalized trace.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let G be a connected semisimple Lie group with finite centre, and let Γ be a lattice in G, that is, Γ is a discrete subgroup of G with finite covolume. Denote by $\hat{G}$ and $\hat{\Gamma}$ the set of equivalence classes of irreducible unitary representations of G and Γ, respectively. A remarkable theorem by M. Cowling and T. Steger asserts that most irreducible unitary representations of G when restricted to Γ remain irreducible and are determined by these restrictions. More precisely, the following holds.

THEOREM A. ([CoS]) *Let π and π' be irreducible unitary representations of G. Assume that π and π' are not square integrable. Then*

(i) *the restriction $\pi \mid \Gamma$ is irreducible*;

(ii) *if $\pi \mid \Gamma$ and $\pi' \mid \Gamma$ are unitarily equivalent*, *then so are π and π'.*

Now $\hat{G}$ and $\hat{\Gamma}$ carry a topology (see below) so that a natural problem is to study the topological properties of the map $\pi \mapsto \pi \mid \Gamma$. For instance, it was shown in [BeV] that, for most lattices Γ, the image of this map is not dense in $\hat{\Gamma}$.

Our aim in this paper is to investigate the spectrum of the restriction to Γ of an irreducible unitary representation π of G as above. We shall also

0022-1236/97 $25.00

study the C^*-algebras associated to such restrictions. This is a family of C^*-algebras generalising the reduced C^*-algebras $C_r^*(\Gamma)$ of Γ. Some previous results were obtained in [BeH] for which this paper may be considered as continuation.

By spectrum of a representation ρ of a locally compact group H, we mean the set of all σ in $\hat{H}$ that are weakly contained in ρ. Recall that σ is weakly contained in $\hat{H}$ if every positive definite function associated with σ is uniform limit over compacta of sums of positive definite functions associated with ρ. Denoting by the same letters the extensions of ρ and σ to the (maximal) C^*-algebra $C^*(H)$ of H, this is equivalent to saying that $C^*\operatorname{Ker}\rho$ is contained in $C^*\operatorname{Ker}\sigma$. As usual, σ and ρ are said to be weakly equivalent if each of them is weakly obtained in the other. When restricted to $\hat{H}$, weak containment defines a topology on $\hat{H}$. This is the weakest topology on $\hat{H}$ for which the map

$$\hat{H} \to \operatorname{Prim} C^*(H), \qquad \pi \mapsto C^*\operatorname{Ker}\pi$$

is continuous, where $\operatorname{Prim} C^*(H)$ is the space of all primitive ideals of $C^*(H)$, equipped with the Jacobson topology. For all this, see [Dix], Chap. 18.

Our first result is a strengthening of Theorem A, (ii), in case G is simple.

THEOREM B. *Let G be a simple Lie group with finite centre, and let Γ be a lattice in G. Let π and π' be irreducible, non-equivalent unitary representations of G. Assume that not both are weakly contained in the regular representation λ_G of G. Let H be a subgroup of G containing Γ. Then the restrictions $\pi \mid H$ and $\pi' \mid H$ are not weakly equivalent, H being equipped with the discrete topology.*

The arguments used in the proof of this theorem yield also a quick proof of the theorem of Cowling and Steger (see Remark 2, below).

It should be noted that the regular representation λ_Γ of Γ is always weakly contained in $\pi \mid \Gamma$ if G has trivial centre and if the (not necessarily irreducible) unitary representation π of G is not a multiple of 1_G, the trivial representation of G (see [BeH], Théorème 1). This shows that, if π and π' are both weakly contained in the regular representation λ_G, then $\pi \mid H$ and $\pi' \mid H$ are weakly equivalent, at least when G has a trivial centre and $H = \Gamma$.

In fact, a much stronger result was proved in [BCH], Theorem 1: assume G has trivial centre, and let H be any Zariski-dense subgroup of G, equipped with the discrete topology. Then the reduced C^*-algebra of H is simple. This means that any unitary representation weakly contained in λ_H is weakly equivalent to λ_H.

It is interesting to reformulate all these results as statements about Prim $C^*(\Gamma)$, a space which is much more tractable than $\hat{\Gamma}$: By simplicity of $C_r^*(\Gamma)$, $\{C^* \operatorname{Ker} \lambda_\Gamma\}$ is a closed point in Prim $C^*(\Gamma)$ and $C^* \operatorname{Ker} \lambda_\Gamma = C^* \operatorname{Ker} \sigma$ for any σ which is weakly contained in λ_Γ. By Theorem B, if π is not weakly contained in λ_G, then $\{C^* \operatorname{Ker} \pi \mid \Gamma\}$ and $\{C^* \operatorname{Ker} \pi' \mid \Gamma\}$ are different points in Prim $C^*(\Gamma)$ for any $\pi' \in \hat{G}$ which is not equivalent to π.

Theorem B has an interesting application to the representation theory of the group of rational points of simple **Q**-algebraic groups.

Corollary 1. *Let* **G** *be a simple algebraic group defined over* **Q**. *Let G be* **G**(**R**), *the group of the real points of* **G**, *and let H be the subgroup* **G**(**Q**) *equipped with the discrete topology. Let π and π' be irreducible, non-equivalent unitary representations of G, and assume that not both are weakly contained in λ_G. Then $\pi \mid H$ and $\pi' \mid H$ are not weakly equivalent.*

Since, by a well-known theorem of Borel and Harish-Chandra, **G**(**Z**) is a lattice in the simple Lie group **G**(**R**), Corollary 1 follows immediately from Theorem B.

Our second result shows that there is a strong restriction on the representations in the spectrum of $\pi \mid \Gamma$ for π in $\hat{G}$. Its proof is based on a property of Γ, established in [BCH], which is crucial for the proof of the simplicity of $C_r^*(\Gamma)$ (see Proposition 2).

Theorem C. *Let G be a simple Lie group with trivial centre, and let Γ be a lattice in G. Let π be an irreducible unitary representation of G, $\pi \neq 1_G$, the trivial one-dimensional representation of G. If σ is any unitary representation of Γ weakly contained in $\pi \mid \Gamma$, then σ weakly contains the regular representation λ_Γ.*

We do not know whether it is possible to replace Γ in Theorem C by any subgroup H containing Γ. The following related result by R. Howe and J. Rosenberg should be mentioned: any unitary representation σ of $PSL(n, \mathbf{Q})$, $\sigma \neq 1_{PSL(n, \mathbf{Q})}$, weakly contains the regular representation $\lambda_{PSL(n, \mathbf{Q})}$ ([HoR], Theorem 2).

Let G, Γ and π in $\hat{G}$ be as in Theorem C. Let $C_\pi^*(\Gamma)$ denote the C^*-subalgebra of $\mathscr{L}(\mathscr{H}_\pi)$ generated by all $\pi(\gamma)$ for γ in Γ, where $\mathscr{L}(\mathscr{H}_\pi)$ is the C^*-algebra of all bounded linear operators on the Hilbert space $\mathscr{H}_\pi$ of π. As we now see, C^*-algebras of this type have some remarkable properties.

First of all, by the theorem of Cowling and Steger quoted above, $C_\pi^*(\Gamma)$ is a primitive algebra for any π in $\hat{G}$ (for square integrable π, see the remark after Theorem B). Observe that Theorem C implies that the regular representation λ_Γ is weakly contained in $\pi \mid \Gamma$ (this fact is also proved in [BeH] as mentioned above). So λ_Γ factorises to a representation $\lambda_{\Gamma, \pi}$ of

$C^*_\pi(\Gamma)$, and hence the reduced C^*-algebra $C^*_r(\Gamma)$ is a quotient of $C^*_\pi(\Gamma)$. Notice also that $C^*_r(\Gamma)=C^*_\pi(\Gamma)$ for any π which is weakly contained in λ_Γ.

The following corollary is an immediate consequence of Theorem C and the simplicity of $C^*_r(\Gamma)$.

COROLLARY 2. *Let G, Γ and π in $\hat{G}$ be as in Theorem C. Then* Ker $\lambda_{\Gamma,\pi}$ *is the unique maximal two-sided ideal of $C^*_\pi(\Gamma)$.*

Recall that a normalized trace on a C^*-algebra A with unit is a linear map $\sigma\colon A\to\mathbf{C}$ such that $\sigma(1)=1$, $\sigma(a^*a)\geqslant 0$ and $\sigma(ab)=\sigma(ba)$ for all a, b in A. It is proved in [BCH], Theorem 1, that the canonical trace τ on $C^*_r(\Gamma)$, defined by $\tau(\lambda_\Gamma(\gamma))=0$ for any γ in $\Gamma\backslash\{0\}$ and $\tau(1)=1$, is the unique normalized trace on $C^*_r(\Gamma)$.

Using arguments similar to that for the proof of Theorem C (which in turn is based on [BCH]), we are able to generalise this to all C^*-algebras $C^*_\pi(\Gamma)$.

THEOREM D. *Let G, Γ and π in $\hat{G}$ be as in Theorem C. Then $\tau\circ\lambda_{\Gamma,\pi}$ is the unique normalized trace on $C^*_\pi(\Gamma)$.*

Remark 1. We do not know which of the C^*-algebras $C^*_\pi(\Gamma)$ are mutually non-isomorphic. However, the following should be mentioned:

(i) If $\pi\in\hat{G}$ is not weakly contained in λ_G, then $C^*_\pi(\Gamma)$ is not isomorphic to $C^*_r(\Gamma)$. Indeed, observe that by Theorem B $\pi\mid\Gamma$ is not weakly contained in λ_Γ and, hence, Ker $\lambda_{\Gamma,\pi}\neq\{0\}$ in this case. The claim now follows from Corollary 2.

(ii) Suppose $\pi,\pi'\in\hat{G}$ are non-equivalent and not both weakly contained in λ_G. Then Theorem B states that there is no isomorphism $T\colon C^*_\pi(\Gamma)\to C^*_{\pi'}(\Gamma)$ such that

$$T(\pi(\gamma))=\pi'(\gamma)\qquad\forall\gamma\in\Gamma.$$

Observe that the result of Cowling and Steger (Theorem A, (ii)) states only that no spatial T with this property exists.

It is a pleasure to thank P. de la Harpe and A. Valette for helpful discussions and comments.

2. PROOF OF THEOREM B

Throughout this section, G denotes a simple non-compact Lie group with finite centre and Γ is a lattice in G.

Let ρ be the quasi-regular representation of G on $L^2(G/\Gamma)$ defined by

$$\rho(g)\,\xi(x)=\xi(g^{-1}x), \qquad \forall g\in G, x\in G/\Gamma, \xi\in L^2(G/\Gamma).$$

Notice that ρ is the unitary representation $\mathrm{Ind}_\Gamma^G(1_\Gamma)$ induced by the trivial representation 1_Γ. Denote by ρ_0 the restriction of ρ to

$$L_0^2(G/\Gamma)=\left\{f\in L^2(G/\Gamma);\int_{G/\Gamma} f(x)\,dx=0\right\},$$

the subspace orthogonal to the constants.

We shall use the following crucial fact which is known to many experts and is proved in [Moo], Proposition 3.6 (see also [BeV], Proposition 2).

Proposition 1. *There exists an integer N such that the N-fold tensor product $\rho_0^{\otimes N}$ is contained in $\infty\lambda_G$, an infinite multiple of the regular representation λ_G of G.*

For G not locally isomorphic to $SO_0(n,1)$ or $SU(n,1)$, the proof of the proposition above follows from results by M. Cowling: indeed, in this case there exists an integer N such that $\pi^{\otimes N}$ is contained in $\infty\lambda_G$ for any unitary representation π of G which does not contain the trivial representation 1_G (see [Cow], Theorems 2.4.2 and 2.5.3). In the remaining cases, the proof relies on the fact that there is a gap over 0 in the spectrum of the Laplacian on the locally symmetric space $K\backslash G/\Gamma$. The existence of such a gap is clear when Γ is cocompact and follows from results by Borel and Garland ([BoG], Theorem 3) in the general case.

To determine the optimal integer N in Proposition 1 is a very interesting and difficult problem. For instance, it is known that $N=1$ in the case $G=SL(2,\mathbf{R})$, $\Gamma=SL(2,\mathbf{Z})$ (see, e.g. [Ter], Section 3.7, Theorem 1) which means that ρ_0 is weakly contained in λ_Γ. It has been conjectured by A. Selberg that the same is true when Γ is any congruence subgroup of $SL(2,\mathbf{Z})$ (this is the famous conjecture $\lambda_1\geqslant 1/4$). For related results in the case $SO(n,1)$, see [EGM], [LPS] and [BuS].

Proof of Theorem B. Let π and π' be non-equivalent irreducible unitary representations of G. Assume that, say, π is not weakly contained in the regular representation λ_G. Suppose, by contradiction, that $\pi\mid H$ and $\pi'\mid H$ are weakly equivalent. Then $\pi\mid\Gamma$ and $\pi'\mid\Gamma$ are weakly equivalent, so the induced representations $\mathrm{Ind}_\Gamma^G\,\pi\mid\Gamma$ and $\mathrm{Ind}_\Gamma^G\,\pi'\mid\Gamma$ are weakly equivalent, by continuity of inducing (see [Fel], Theorem 4.1). But

$$\mathrm{Ind}_\Gamma^G\,\pi\mid\Gamma=\pi\otimes\mathrm{Ind}_\Gamma^G\,1_\Gamma=\pi\oplus(\pi\otimes\rho_0),$$

and similarly $\mathrm{Ind}_\Gamma^G\,\pi'\mid\Gamma=\pi'\oplus(\pi'\otimes\rho_0)$. Observe that π and π' are not weakly equivalent. Indeed, G is a group of type I so that $\mathrm{Ker}\,\pi\neq\mathrm{Ker}\,\pi'$

since π and π' are not equivalent (see [Dix], Theorem 9.1). By irreducibility of π and π', this implies that π is weakly contained in $\pi' \otimes \rho_0$ and that π' is weakly contained in $\pi \otimes \rho_0$. Therefore, π is weakly contained in

$$(\pi \otimes \rho_0) \otimes \rho_0 = \pi \otimes \rho_0^{\otimes 2}.$$

By induction, we see that π is weakly contained in $\pi \otimes \rho_0^{\otimes 2n}$ for any $n \in \mathbf{N}$. Hence, from the proposition above we deduce that π is weakly contained in the regular representation λ_G. This is a contradiction, and the proof is complete. ∎

Remark 2. Essentially the same argument gives a short proof for the theorem of Cowling and Steger (Theorem A). For instance, let us show that $\pi \mid \Gamma$ is irreducible if $\pi \in \hat{G}$ is not square integrable. By an elementary lemma (see [CoS], Corollaries 1.2 and 1.3), it is sufficient to show that π is not contained in $\pi \otimes \rho_0$. But if this were the case, then π would be contained in $\pi \otimes \rho_0^{\otimes 2}$. In this way, we see that π would be contained in $\pi \otimes \rho_0^{\otimes N}$ and hence in $\infty \lambda_G$ by the proposition above. This is a contradiction as π is not square integrable.

Remark 3. In the case $G = PSL(2, \mathbf{R})$, more precise results were obtained in [BeH]:

Let $(\pi_t)_{0<t<1}$ be the complementary series of $PSL(2, \mathbf{R})$ parametrized as in [Lan]. Then, for any lattice Γ, $\pi_t \mid \Gamma$ is not weakly contained in $\pi_s \mid \Gamma$ for $s < t$. Moreover, if $\Gamma = PSL(2, \mathbf{Z})$, then the map

$$]0, 1[\rightarrow \hat{\Gamma}, \qquad t \mapsto \pi_t \mid \Gamma$$

is a homeomorphism onto its image ([BeH], Théorème 2 and Théorème 3).

Remark 4. The following amazing fact about the ideal theory of $l^1(\Gamma)$ should be mentioned.

Let $\mathrm{Prim}_* l^1(\Gamma)$ be the set of all kernels of irreducible *-representations of $l^1(\Gamma)$. By the theorem of Cowling and Steger (and the simplicity of $C_r^*(\Gamma)$), $l^1 \operatorname{Ker} \pi \mid \Gamma = C^* \operatorname{Ker} \pi \mid \Gamma \cap l^1(\Gamma)$ is in $\mathrm{Prim}_* l^1(\Gamma)$ for any $\pi \in \hat{G}$, $\pi \neq 1_G$. But since λ_Γ is weakly contained in $\pi \mid \Gamma$, $\pi \mid \Gamma$ is faithful on $l^1(\Gamma)$. Hence, $l^1 \operatorname{Ker} \pi \mid \Gamma = \{0\}$ for all $\pi \in \hat{G}$, $\pi \neq 1_G$. On the other hand, by Theorem B, $C^* \operatorname{Ker} \pi \mid \Gamma \neq C^* \operatorname{Ker} \pi \mid \Gamma'$ if π and π' are not equivalent and not both weakly contained in λ_Γ. We thus see that the map

$$\mathrm{Prim}\, C^*(\Gamma) \rightarrow \mathrm{Prim}_* l^1(\Gamma), \qquad P \mapsto P \cap l^1(\Gamma)$$

is far from being injective. This is in sharp contrast to the case of connected Lie group where the above map is often injective (see [Boi], [Pog]).

3. PROOFS OF THEOREM C AND THEOREM D

In this section, G denotes a non-compact simple Lie group with trivial centre and Γ a lattice in G.

The following result, which is proved in [BCH] (Lemma 2.3, Lemma 2.4 and Theorem 2), is the crucial step in proving the simplicity of the C^*-algebra $C_r^*(\Gamma)$.

PROPOSITION 2. ([*BCH*]) *Let F be a finite subset of $\Gamma\backslash\{1\}$. Then there exist y_0 in Γ and a constant C such that*

$$\left\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leqslant C \|a\|_2 \qquad \forall a \in l^2(\mathbf{N}), \quad \forall x \in F,$$

where a_j is the j th term of the sequence a.

The proofs of Theorem C and Theorem D depend on the following consequence of Proposition 2.

LEMMA. *Let π be an irreducible unitary representation of G, $\pi \neq 1_G$. Let σ be a unitary representation of Γ, weakly contained in $\pi \mid \Gamma$. Let F be a finite subset of $\Gamma\backslash\{1\}$, and let y_0 in Γ and $C>0$ be as in Proposition 2. Then there is a real number p, $1 \leqslant p < \infty$, with the following property: if φ is any matrix coefficient of σ, then the sequence $(\varphi(y_0^{-j} x y_0^j))_{j \geqslant 1}$ lies in $l^p(\mathbf{N})$ for all $x \in F$.*

Proof. It is well-known that there is a real number q, $1 \leqslant q < \infty$, such that all the K-finite matrix coefficients of π lie in $L^q(G)$, K being a maximal compact subgroup of G (see, e.g., [BoW], VI, 5.4.). Hence, there is some integer N such that the N-fold tensor product $\pi^{\otimes N}$ is weakly contained in the regular representation λ_G. Since $\sigma^{\otimes N}$ is weakly contained in $\pi^{\otimes N} \mid \Gamma$, this implies that $\sigma^{\otimes N}$ is weakly contained in λ_Γ. This means that

$$\|\sigma^{\otimes N}(f)\| \leqslant \|\lambda_\Gamma(f)\| \qquad \forall f \in l^1(\Gamma).$$

Therefore, by Proposition 2, we have

$$\left\| \sum_{j=1}^{\infty} a_j \sigma^{\otimes N}(y_0^{-j} x y_0^j) \right\| \leqslant C \|a\|_2 \qquad \forall a \in l^2(\mathbf{N}), \quad \forall x \in F.$$

Let $\varphi = \langle \sigma(\cdot)\, \xi, \eta \rangle$ be a matrix coefficient of σ. Then φ^N is a matrix coefficient of $\sigma^{\otimes N}$, and the inequality above implies that

$$\left| \sum_{j=1}^{\infty} a_j \varphi^N(y_0^{-j} x y_0^j) \right| \leqslant C \|\xi\|^N \|\eta\|^N \|a\|_2 \qquad \forall a \in l^2(\mathbf{N}), \quad \forall x \in F.$$

Hence, the sequence $(\varphi^N(y_0^{-j}xy_0^j))_{j\geqslant 1}$ is in $l^2(\mathbf{N})$, and the lemma is proved. ▮

Proof of Theorem C. Let σ be a unitary representation of Γ, weakly contained in $\pi \mid \Gamma$.

Since λ_Γ is a cyclic representation, it suffices to prove that δ_1, the Dirac function at the group unit 1, is uniform limit over finite subsets of Γ of sums of positive definite functions associated with σ.

Let F be a finite subset of $\Gamma\backslash\{1\}$, and let y_0 be as in Proposition 2. Let ξ be a unit vector in the Hilbert space of σ, and define

$$\varphi(\gamma) = \langle \sigma(\gamma)\, \xi, \xi \rangle \qquad \forall \gamma \in \Gamma.$$

Then φ is a positive definite function associated with σ, and it follows from the lemma above that

$$\lim_{j\to\infty} \varphi(y_0^{-1}xy_0^j) = 0 \qquad \forall x \in F.$$

Thus,

$$\lim_{j\to\infty} \langle \sigma(x)\, \sigma(y_0^j)\, \xi, \sigma(y_0^j)\, \xi \rangle = \delta_1(x) \qquad \forall x \in F \cup \{1\}. \quad ▮$$

Proof of Theorem D. Let $\tilde{\tau}$ a normalized trace on $C^*_\pi(\Gamma)$. The function $\varphi\colon \Gamma \to \mathbf{C}$, defined by

$$\varphi(\gamma) = \tilde{\tau}(\pi(\gamma)) \qquad \forall \gamma \in \Gamma$$

is a positive definite function on Γ, associated to a representation σ which is weakly contained in $\pi \mid \Gamma$.

Let x be a group element in $\Gamma\backslash\{1\}$, and let y_0 be as in Proposition 2, applied to $F = \{x\}$. Then, by the lemma above,

$$\lim_{j\to\infty} \varphi(y_0^{-j}xy_0^j) = 0.$$

Hence, $\varphi(x) = 0$ since φ is invariant under conjugation. By continuity, this clearly implies that $\tilde{\tau} = \tau$. ▮

REFERENCES

[BCH] M. Bekka, M. Cowling, and P. de la Harpe, Some groups whose reduced C^*-algebra is simple, *Publ. Math. IHES* **80** (1994), 117–134.

[BeH] M. Bekka and P. de la Harpe, Représentations d'un groupe faiblement équivalentes à la représentation régulière, *Bull. Soc. Math. France* **122** (1994), 333–342.

[BeV] M. Bekka and A. Valette, Lattices in semi-simple Lie groups, and multipliers of group C^*-algebras, *in* "Proceedings of ALGOP, Orléans, July 1992," *Astérique* **232** (1995), 67–79.

[BoG] A. Borel and H. Garland, Laplacian and the discrete spectrum of an arithmetic group, *Amer. J. Math.* **105** (1983), 309–335.

[Boi] J. Boidol, H. Leptin, J. Schürmann, and D. Vahle, Räume primitiver Ideale von Gruppenalgebren, *Math. Ann.* **236** (1978), 1–13.

[BoW] A. Borel and N. Wallach, Continuous cohomology, discrete groups, and representations of reductive groups, *Ann. of Math. Stud.* **94** (1980).

[BuS] M. Burger and P. Sarnak, Ramanujan duals II, *Invent. Math.* **106**, 1–11.

[CoS] M. Cowling and T. Steger, The irreducibility of restrictions of unitary representations to lattices, *J. Reine Angew. Math.* **420** (1991), 85–98.

[Cow] M. Cowling, Sur les coefficients des représentations unitaires des groupes de Lie simples, *in* "Analyse Harmonique," *Lect. Notes Math.* **739** (1979), 132–178.

[Dix] J. Dixmier, "C^*-Algebras," North-Holland, Amsterdam, 1977.

[EGM] J. Elstrodt, F. Grunewald, and J. Mennicke, Kloosterman sums for Clifford algebras and a lower bound for the positive eigenvalues of the laplacian for congruence subgroups acting on hyperbolic spaces, *Invent. Math.* **101** (1990), 641–685.

[Fel] J. M. G. Fell, Weak containment and induced representations of groups, *Canad. J. Math.* **14** (1962), 237–268.

[Har] P. de la Harpe, Operator algebras, free groups and other groups, *in* "Proceedings of ALGOP, Orléans, July 1992," *Astérique* **232** (1995), 121–153.

[HoR] R. E. Howe and J. Rosenberg, The unitary representation theory of $GL(n)$ of an infinite discrete field, *Israel J. Math.* **67** (1989), 67–81.

[Lan] S. Lang, "$SL(2, \mathbf{R})$," Addison–Wesley, Reading, MA, 1975.

[LPS] J. S. Li, I. I. Piatetskii-Shapiro, and P. Sarnak, Poincaré series for $SO(n, 1)$, *Proc. Indian Acad. Sci. Math. Sci.* **97** (1987), 231–237.

[Moo] C. C. Moore, Exponential decay of correlations coefficients for geodesic flows, *in* "Group Representations, Ergodic Theory, Operator Algebras, and Mathematical Physics," pp. 163–181, Proc. Conf. in honor of G. W. Mackey, MSRI Publications, Springer, 1987.

[Pog] D. Poguntke, Unitary representations of Lie groups and operators of finite rank, *Ann. Math.* **140** (1994), 503–556.

[Ter] A. Terras, "Harmonic Analysis on Symmetric Spaces," Springer, Berlin, 1985.